HERMITIAN CATEGORIES, EXTENSION OF SCALARS AND SYSTEMS OF SESQUILINEAR FORMS

EVA BAYER-FLUCKIGER[1], URIYA A. FIRST[2], AND DANIEL A. MOLDOVAN[1]

ABSTRACT. We prove that the category of *systems of sesquilinear forms* over a given hermitian category is equivalent to the category of *unimodular* 1-*hermitian forms* over another hermitian category. The sesquilinear forms are not required to be unimodular or defined on a reflexive object (i.e. the standard map from the object to its double dual is not assumed to be bijective), and the forms in the system can be defined with respect to different hermitian structures on the given category. This extends a result obtained in [3].

We use the equivalence to define a Witt ring of sesquilinear forms over a hermitian category, and also to generalize various results (e.g.: Witt's Cancelation Theorem, Springer's Theorem, the weak Hasse principle, finiteness of genus) to systems of sesquilinear forms over hermitian categories.

INTRODUCTION

Quadratic and hermitian forms were studied extensively by various authors, who have developed a rich array of tools to study them. It is well-known that in many cases (e.g. over fields), the theory of sesquilinear forms can be reduced to the theory of hermitian forms (e.g. see [14], [13] and works based on them). In the recent paper [3], an explanation of this reduction was provided in the form of an equivalence between the category of sesquilinear forms over a ring and the category of unimodular 1-hermitian forms over a special hermitian category.

In this paper, we extend the equivalence of [3] to hermitian categories, and moreover, improve it in such a way that it applies to systems of sesquilinear forms in hermitian categories that admit *non-reflexive* objects (see section 2). That is, we prove that the category of systems of sesquilinear forms over a hermitian category $\mathscr{C}$ is equivalent to the category of unimodular 1-hermitian forms over anther hermitian category $\mathscr{C}'$. The sesquilinear forms are not required to be

unimodular or defined on a reflexive object, and the forms in the system can be defined with respect to different hermitian structures on the category $\mathscr{C}$.

Using the equivalence, we present a notion of a Witt group of sesquilinear forms, which is analogous to the standard Witt group of hermitian forms over rings with involution (e.g. see [10] or [16]). We also extend various results (Witt's Cancelation Theorem, Springer's Theorem, finiteness of genus, the Hasse principle, etc.) to systems of sesquilinear forms over hermitian categories (and in particular to systems of sesquilinear forms over rings with a family of involutions).

Sections 1 and 2 recall the basics of sesquilinear forms over rings and hermitian categories, respectively. In section 3, we prove the equivalence of the category of sesquilinear forms over a given hermitian category to a category of unimodular 1-hermitian forms over another hermitian category, and in section 4, we extend this result to systems of sesquilinear forms. Section 5 presents applications of the equivalence.

1. Sesquilinear and Hermitian Forms

Let A be a ring. An *involution* on A is an additive map $\sigma : A \to A$ such that $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$ and $\sigma^2 = \mathrm{id}_A$. Let V be a right A-module. A *sesquilinear form* over (A, σ) is a biadditive map $s : V \times V \to A$ satisfying $s(xa, yb) = \sigma(a)s(x, y)b$ for all $x, y \in V$ and $a, b \in A$. The pair (V, s) is also called a sesquilinear form in this case.[1] The *orthogonal sum* of two sesquilinear forms (V, s) and (V', s') is defined to be $(V \oplus V', s \oplus s')$ where $s \oplus s'$ is given by

$$(s \oplus s')(x \oplus x', y \oplus y') = s(x, y) + s'(x', y')$$

for all $x, y \in V$ and $x', y' \in V'$. Two sesquilinear forms (V, s) and (V', s') are called *isometric* if there exists an isomorphism of A-modules $f : V \xrightarrow{\sim} V'$ such that $s'(f(x), f(y)) = s(x, y)$ for all $x, y \in V$.

Let $V^* = \mathrm{Hom}_A(V, A)$. Then V^* has a right A-module structure given by $(f \cdot a)(x) = \sigma(a)f(x)$ for all $f \in V^*$, $a \in A$. We say that V is *reflexive* if the homomorphism of right A-modules $\omega_V : V \to V^{**}$ defined by $\omega_V(x)(f) = \sigma(f(x))$ for all $x \in V$, $f \in V^*$ is bijective.

A sesquilinear space (V, s) over (A, σ) induces two homomorphisms of right A-modules $s_\ell, s_r : V \to V^*$ called the *left* and *right adjoint* of s, respectively. They are given by $s_\ell(x)(y) = s(x, y)$ and $s_r(x)(y) = \sigma(s(y, x))$ for all $x, y \in V$. Observe that $s_r = s_\ell^* \omega_V$ and $s_\ell = s_r^* \omega_V$. The form s is called *unimodular* if s_r and s_ℓ are isomorphisms. In this case, V must be reflexive.

Let $\epsilon = \pm 1$. A sesquilinear form (V, s) over (A, σ) is called *ϵ-hermitian* if $\sigma(s(x, y)) = \epsilon s(y, x)$ for all $x, y \in V$, i.e. $s_r = \epsilon s_\ell$. A 1-hermitian form is also called a hermitian form.

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[1] Some texts use the term *sesquilinear space.*

There exists a classical notion of Witt group for unimodular ϵ-hermitian forms over (A, σ) (e.g. see [10]): Denote by $\mathrm{WG}^\epsilon(A, \sigma)$ the Grothendieck group of isometry classes of unimodular ϵ-hermitian forms (V, s) over (A, σ) with V finite projective, the addition being orthogonal sum. A unimodular ϵ-hermitian form over (A, σ) is called *hyperbolic* if it is isometric to $(\mathbb{H}_V^\epsilon, V \oplus V^*)$ for some f.g. projective right A-module V, where $\mathbb{H}_V^\epsilon$ is defined by

$$\mathbb{H}_V^\epsilon(x \oplus f, y \oplus g) = f(y) + \epsilon\sigma(g(x)) \qquad \forall x, y \in V, \ f, g \in V^* .$$

We let $\mathbb{H}_V = \mathbb{H}_V^1$. The quotient of $\mathrm{WG}^\epsilon(A, \sigma)$ by the subgroup generated by the unimodular ϵ-hermitian hyperbolic forms is called the *Witt group of unimodular ϵ-hermitian forms* over (A, σ) and is denoted by $\mathrm{W}^\epsilon(A, \sigma)$.

We denote by $\mathrm{Sesq}(A, \sigma)$ (resp. $\mathrm{UH}^\epsilon(A, \sigma)$) the category of sesquilinear (resp. unimodular ϵ-hermitian) forms over (A, σ). The morphisms of these categories are (bijective) isometries. For simplicity, we let $\mathrm{UH}(A, \sigma) := \mathrm{UH}^1(A, \sigma)$.

2. Hermitian Categories

This section recalls some basic notions about hermitian categories as presented in [16] (see also [10], [11]).

2.1. **Preliminaries.** Recall that a *hermitian category* consists of a triplet $(\mathscr{C}, *, \omega)$ where $\mathscr{C}$ is an additive category, $* : \mathscr{C} \to \mathscr{C}$ is a contravariant functor and $\omega = (\omega_C)_{C \in \mathscr{C}} : \mathrm{id} \to **$ is a natural transformation satisfying $\omega_C^* \omega_{C^*} = \mathrm{id}_{C^*}$ for all $C \in \mathscr{C}$. In this case, the pair $(*, \omega)$ is called a *hermitian structure* on $\mathscr{C}$. It is customary to assume that ω is a natural *isomorphism* rather than a natural *transformation*. Such hermitian categories will be called *reflexive*. In general, an object $C \in \mathscr{C}$ for which ω_C is an isomorphism is called *reflexive*, so the category $\mathscr{C}$ is reflexive precisely when all its objects are reflexive. We will often drop $*$ and ω from the notation and use these symbols to denote the functor and natural transformation associated with any hermitian category under discussion.

A *sesquilinear form* over the category $\mathscr{C}$ is a pair (C, s) with $C \in \mathscr{C}$ and $s : C \to C^*$. A sesquilinear form (C, s) is called *unimodular* if s and $s^* \omega_C$ are isomorphisms. (If C is reflexive, then s is bijective if and only if $s^* \omega_C$ is bijective.) Let $\epsilon = \pm 1$. A sesquilinear form (C, s) is called ϵ-*hermitian* if $s = \epsilon s^* \omega_C$. For brevity, 1-hermitian forms are often called hermitian forms. Orthogonal sums of forms are defined in the obvious way. Let (C, s) and (C', s') be two sesquilinear forms over $\mathscr{C}$. An *isometry* from (C, s) to (C', s') is an isomorphism $f : C \xrightarrow{\sim} C'$ satisfying $s = f^* s' f$. In this case, (C, s) and (C', s') are said to be *isometric*. We let $\mathrm{Sesq}(\mathscr{C})$ stand for the category of sesquilinear forms over $\mathscr{C}$ with isometries as morphisms.

Denote by $\mathrm{UH}^\epsilon(\mathscr{C})$ the category of unimodular ϵ-hermitian forms over $\mathscr{C}$. The morphisms are isometries. For brevity, let $\mathrm{UH}(\mathscr{C}) := \mathrm{UH}^1(\mathscr{C})$. The hyperbolic

unimodular ϵ-hermitian forms over $\mathscr{C}$ are the forms isometric to $(\mathbb{H}_Q^\epsilon, Q \oplus Q^*)$, where Q is any reflexive object in $\mathscr{C}$ and $\mathbb{H}_Q^\epsilon$ is given by

$$\mathbb{H}_Q^\epsilon = \begin{bmatrix} 0 & id_{Q^*} \\ \epsilon\omega_Q & 0 \end{bmatrix} : Q \oplus Q^* \to (Q \oplus Q^*)^* = Q^* \oplus Q^{**}.$$

Again, let $\mathbb{H}_Q = \mathbb{H}_Q^1$. The quotient of $\mathrm{WG}^\epsilon(\mathscr{C})$, the Grothendieck group of isometry classes of unimodular ϵ-hermitian forms over $\mathscr{C}$ (w.r.t. orthogonal sum), by the subgroup generated by the hyperbolic forms is called the Witt group of unimodular ϵ-hermitian forms over $\mathscr{C}$ and is denoted by $\mathrm{W}^\epsilon(\mathscr{C})$. For brevity, set $\mathrm{W}(\mathscr{C}) = \mathrm{W}^1(\mathscr{C})$.

Example 2.1. Let (A, σ) be a ring with involution. If we take $\mathscr{C}$ to be Mod-A, the category of right A-modules, and define $*$ and ω as in section 1, then $\mathscr{C}$ becomes a hermitian category. Furthermore, the sesquilinear forms (M, s) over (A, σ) correspond to the sesquilinear forms over $\mathscr{C}$ via $(M, s) \mapsto (M, s_r)$. This correspondence gives rise to isomorphisms of categories $\mathrm{Sesq}(A, \sigma) \cong \mathrm{Sesq}(\mathscr{C})$ and $\mathrm{UH}^\epsilon(A, \sigma) \cong \mathrm{UH}^\epsilon(\mathscr{C})$. Now let $\mathscr{C}$ be a subcategory of Mod-A such that $M \in \mathscr{C}$ implies $M^* \in \mathscr{C}$. Then $\mathscr{C}$ is still a hermitian category and it is reflexive if and only if $\mathscr{C}$ consists of reflexive A-modules (as defined in section 1). For example, this happens if $\mathscr{C} = \mathcal{P}(A)$, the category of projective A-modules of finite type. In this case, the Witt group $\mathrm{W}^\epsilon(\mathscr{C}) = \mathrm{W}^\epsilon(\mathcal{P}(A))$ is isomorphic to $\mathrm{W}^\epsilon(A, \sigma)$.

2.2. **Duality Preserving Functors.** Let $\mathscr{C}$ and $\mathscr{C}'$ be two hermitian categories. A *duality preserving functor* from $\mathscr{C}$ to $\mathscr{C}'$ is an additive functor $F : \mathscr{C} \to \mathscr{C}'$ together with a natural isomorphism $i = (i_M)_{M \in \mathscr{C}} : F* \to *F$. This means that for any $M \in \mathscr{C}$, there exists an isomorphism $i_M : F(M^*) \xrightarrow{\sim} (FM)^*$ such that for all $N \in \mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(M, N)$, the following diagram commutes:

$$\begin{array}{ccc} F(N^*) & \xrightarrow{F(f^*)} & F(M^*) \\ \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle i_M} \\ (FN)^* & \xrightarrow{(Ff)^*} & (FM)^* \end{array}$$

Any duality preserving functor induces a functor $\mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}')$, which we also denote by F. It is given by

$$F(M, s) = (FM, i_M F(s))$$

for every $(M, s) \in \mathrm{Sesq}(\mathscr{C})$. If the functor $F : \mathscr{C} \to \mathscr{C}'$ is faithful (resp. faithful and full, induces an equivalence), then so is the functor $F : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}')$.

Let $\lambda = \pm 1$. A duality preserving functor F is called λ-*hermitian* if

$$i_{M^*} F(\omega_M) = \lambda i_M^* \omega_{FM}$$

for all $M \in \mathscr{C}$. Let $\epsilon = \pm 1$. We recall from [10, pp. 80-81] that in this case, the functor $F : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}')$ maps $\mathrm{UH}^\epsilon(\mathscr{C})$ to $\mathrm{UH}^{\epsilon\lambda}(\mathscr{C}')$ and sends ϵ-hermitian hyperbolic forms to $\epsilon\lambda$-hermitian hyperbolic forms. Therefore, F induces a homomorphism between the corresponding Witt groups:

$$\mathrm{W}^\epsilon(F) : \mathrm{W}^\epsilon(\mathscr{C}) \to \mathrm{W}^{\epsilon\lambda}(\mathscr{C}').$$

If F is an equivalence of categories, then $\mathrm{Sesq}(F) : \mathrm{UH}^\epsilon(\mathscr{C}) \to \mathrm{UH}^{\epsilon\lambda}(\mathscr{C}')$ is also an equivalence of categories and the induced group homomorphism $\mathrm{W}^\epsilon(F)$ is an isomorphism of groups.

2.3. **Transfer into the Endomorphism Ring.** The aim of this subsection is to introduce the method of *transfer into the endomorphism ring*, which allows us to pass from the abstract setting of hermitian categories to that of a ring with involution, which is more concrete. This method will be applied repeatedly in section 5. Note that it applies well only to reflexive hermitian categories.

Let $\mathscr{C}$ be a *reflexive* hermitian category, and let M be an object of $\mathscr{C}$, on which we suppose that there exists a unimodular ϵ_0-hermitian form h_0 for a certain $\epsilon_0 = \pm 1$. Denote by E the endomorphism ring of M. According to [11, Lm. 1.2], the form (M, h_0) induces on E an involution σ, defined by $\sigma(f) = h_0^{-1} f^* h_0$ for all $f \in E$. Let $\mathcal{P}(E)$ denote the category of projective right E-modules of finite type. Then, using σ, we can consider $\mathcal{P}(E)$ as a reflexive hermitian category (see Example 2.1).

Recall that an *idempotent* $e \in \mathrm{End}_{\mathscr{C}}(M)$ *splits* if there exist an object $M' \in \mathscr{C}$ and morphisms $i : M' \to M$, $j : M \to M'$ such that $ji = \mathrm{id}_{M'}$ and $ij = e$.

Denote by $\mathscr{C}|_M$ the full subcategory of $\mathscr{C}$ consisting of objects of $\mathscr{C}$ which are isomorphic to a direct summand of a finite direct sum of copies of M. We consider the following functor:

$$\mathrm{T} = \mathrm{T}_{(M,h_0)} := \mathrm{Hom}(M, \_) : \mathscr{C}|_M \to \mathcal{P}(E)$$

$$N \mapsto \mathrm{Hom}(M, N), \qquad \forall N \in \mathscr{C}|_M$$

$$f \mapsto \mathrm{T}(f), \qquad \forall f \in \mathrm{Hom}(N, N'), \ \forall N, N' \in \mathscr{C}|_M,$$

where for all $g \in \mathrm{Hom}(M, N)$, $\mathrm{T}(f)(g) = fg$. In [11, Pr. 2.4], it has been proven the functor T is fully faithful and duality preserving with respect to the natural isomorphism $i = (i_N)_{N \in \mathscr{C}|_M} : \mathrm{T}* \to *\mathrm{T}$ given by $i_N(f) = \mathrm{T}(h_0^{-1} f^* \omega_N)$ for every $N \in \mathscr{C}|_M$ and $f \in \mathrm{Hom}(M, N^*)$. In addition, if all the idempotents of $\mathscr{C}|_M$ split, then T is an equivalence of categories. By computation, we easily see that T is ϵ_0-hermitian.

Note that for any finite list of (reflexive) objects $M_1, \ldots, M_t \in \mathscr{C}$ and any $\epsilon_0 = \pm 1$, there exists a unimodular ϵ_0-hermitian form (M, h_0) such that $M_1, \ldots, M_t \in \mathscr{C}|_M$. Indeed, let $N = \bigoplus_{i=1}^t M_i$ and take $(M, h_0) = (N \oplus N^*, \mathbb{H}_N^{\epsilon_0})$. This means that as long as we treat finitely many hermitian forms, we may pass to the context of hermitian forms over rings with involution.

2.4. **Linear Hermitian Categories and Ring Extension.** In this subsection we introduce the notion of extension of rings in hermitian categories.

Let K be a commutative ring. Recall that a K-*category* is an additive category $\mathscr{C}$ such that for every $A, B \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is endowed with a K-module structure such that the composition is K-bilinear. For example, any additive category is in fact a $\mathbb{Z}$-category. An additive covariant functor $F : \mathscr{C} \to \mathscr{C}'$ between two K-categories is K-*linear* if the map $F : \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}'}(FA, FB)$ is K-linear for all $A, B \in \mathscr{C}$. K-linear contravariant functors are defined in the same manner. A K-*linear hermitian category* is a hermitian category $(\mathscr{C}, *, \omega)$ such that $\mathscr{C}$ is a K-category and $*$ is K-linear.

Fix a commutative ring K. Let $\mathscr{C}$ be an additive K-category and let R be a K-algebra (with unity, not necessarily commutative). We define the *extension of the category $\mathscr{C}$ to the ring R*, denoted $\mathscr{C} \otimes_K R$, to be the category whose objects are formal symbols $C \otimes_K R$ with $C \in \mathscr{C}$ and its Hom-sets are defined by

$$\mathrm{Hom}_{\mathscr{C} \otimes_K R}(A \otimes_K R, B \otimes_K R) = \mathrm{Hom}_{\mathscr{C}}(A, B) \otimes_C R .$$

The composition in $\mathscr{C} \otimes_K R$ is defined in the obvious way. It is straightforward to check that $\mathscr{C} \otimes_K R$ is also a K-category. Moreover, when R is commutative, $\mathscr{C} \otimes_K R$ is an R-category. We define the *scalar extension functor*, $\mathcal{R}_{R/K} : \mathscr{C} \to \mathscr{C} \otimes_K R$ by

$$\mathcal{R}_{R/K} M = M \otimes_K R, \qquad \forall M \in \mathscr{C} \qquad \text{and}$$
$$\mathcal{R}_{R/K} f = f \otimes_K 1, \qquad \forall f \in \mathrm{Hom}(M, N).$$

The functor $\mathcal{R}_{R/K}$ is additive and K-linear.

In case K is obvious from context, we write $\mathscr{C}_R$, M_R, f_R instead of $\mathscr{C} \otimes_K R$, $M \otimes_K R$, $f \otimes_K 1$, respectively. (Here, $M \in \mathscr{C}$ and f is a morphism in $\mathscr{C}$.)

Remark 2.2. The scalar extension we have just defined agrees with scalar extension of modules under mild assumptions ,but not in general: Let S and R be two K-algebras and write $S_R = S \otimes_K R$. There is an additive functor $G : (\mathrm{Mod}\text{-}S)_R \to \mathrm{Mod}\text{-}(S_R)$ given by

$$G(M_R) = M \otimes_S S_R \qquad \text{and}$$
$$G(f \otimes a)(m \otimes b) = fm \otimes ab$$

for all $M, N \in \mathrm{Mod}\text{-}S$, $f \in \mathrm{Hom}_S(M, N)$, $a, b \in R$, and the following diagram commutes

$$
\begin{array}{ccc}
\mathrm{Mod}\text{-}S & \xrightarrow{\ \mathcal{R}_{R/K}\ } & (\mathrm{Mod}\text{-}S)_R \\
\Big\| & & \Big\downarrow {\scriptstyle G} \\
\mathrm{Mod}\text{-}S & \xrightarrow{\ \ -\otimes_S S_R\ \ } & \mathrm{Mod}\text{-}(S_R)
\end{array}
$$

In general, G is neither full nor faithful. However, using standard tensor-Hom relations, it is easy to verify that the map

(1) $$G : \mathrm{Hom}_{(\mathrm{Mod}\text{-}S)_R}(M_R, M'_R) \to \mathrm{Hom}_{\mathrm{Mod}\text{-}(S_R)}(GM_R, GM'_R)$$

is bijective if either (a) M is projective of finite type or (b) R is a flat K-module and M is finitely presented. In particular, if $\mathscr{C}$ is an additive subcategory of Mod-S consisting of f.p. modules and R is flat as a K-module, then $\mathscr{C}_R$ can be understood as a full subcategory of Mod-(S_R) in the obvious way. An example in which the map G of (1) is neither injective nor surjective can be obtained by taking $S = K = \mathbb{Z}$, $R = \mathbb{Q}$ and $M = M' = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

If $(\mathscr{C}, *, \omega)$ is a K-linear hermitian category and R/K is a *commutative* ring extension, then $\mathscr{C}_R$ also has a hermitian structure given by $(M_R)^* = (M^*)_R$, $(f \otimes a)^* = f^* \otimes a$ and $\omega_{M_R} = (\omega_M)_R = \omega_M \otimes 1$ for all $M, N \in \mathscr{C}$, $f \in \mathrm{Hom}_{\mathscr{C}}(M, N)$ and $a \in R$. In this case, the functor $\mathcal{R}_{R/K}$ is a 1-hermitian duality preserving functor (the natural transformation $i : \mathcal{R}_{R/K}* \to *\mathcal{R}_{R/K}$ is just the identity). In particular, we get a functor $\mathcal{R}_{R/K} : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{Sesq}(\mathscr{C}_R)$ given by $\mathcal{R}_{R/K}(M, s) := (M_R, s_R)$ and $\mathcal{R}_{R/K}$ sends ϵ-hermitian (hyperbolic) forms to ϵ-hermitian (hyperbolic) forms.

2.5. **Scalar Extension Commutes with Transfer.** Let R/K be a commutative ring extension, let $\mathscr{C}$ be a reflexive K-linear hermitian category and let M be an object of $\mathscr{C}$ admitting a unimodular ϵ-hermitian form h. Then (M_R, h_R) is a unimodular ϵ-hermitian form over $\mathscr{C}_R$. Let $E = \mathrm{End}_{\mathscr{C}}(M)$ and $E_R = \mathrm{End}_{\mathscr{C}_R}(M_R) = E \otimes_K R$. It is easy to verify that the following diagram (of functors) commutes

$$
\begin{array}{ccc}
\mathscr{C}|_M & \xrightarrow{\mathrm{T}_{(M,h)}} & \mathcal{P}(E) \\
\downarrow{\scriptstyle \mathcal{R}_{R/K}} & & \downarrow{\scriptstyle \_ \otimes_E E_R} \\
\mathscr{C}_R|_{M_R} & \xrightarrow{\mathrm{T}_{(M_R, h_R)}} & \mathcal{P}(E_R)
\end{array} \quad .
$$

(Note that by Remark 2.2, $\mathcal{P}(E_R)$ and $\_ \otimes_E E_R$ can be understood as $\mathcal{P}(E)_R$ and $\mathcal{R}_{R/K}$, respectively.) Since all the functors are ϵ- or 1-hermitian, we get the following commutative diagram, in which the horizontal arrows are full and faithful,

$$
\begin{array}{ccc}
\mathrm{UH}^{\lambda}(\mathscr{C}|_M) & \xrightarrow{\mathrm{T}_{(M,h)}} & \mathrm{UH}^{\lambda\epsilon}(\mathcal{P}(E)) \\
\downarrow{\scriptstyle \mathcal{R}_{R/K}} & & \downarrow{\scriptstyle \_ \otimes_E E_R} \\
\mathrm{UH}^{\lambda}(\mathscr{C}_R|_{M_R}) & \xrightarrow{\mathrm{T}_{(M_R, h_R)}} & \mathrm{UH}^{\lambda\epsilon}(\mathcal{P}(E_R))
\end{array} \quad .
$$

This diagram means that in order to study the behavior of $\mathcal{R}_{R/K}$ on arbitrary K-linear hermitian categories, it is enough to study its behavior on hermitian categories obtained from K-algebras with K-involution (as in Example 2.1).

3. An Equivalence of Categories

Let $\mathscr{C}$ be a (not-necessarily reflexive) hermitian category. In this section we prove that there exists a *reflexive* hermitian category $\mathscr{C}'$ such that the category

$\mathrm{Sesq}(\mathscr{C})$ is equivalent to $\mathrm{UH}^1(\mathscr{C}')$. (We explain how to extend this result to *systems of sesquilinear forms* in the next section.)

The category $\mathscr{C}'$ resembles the category of double arrows presented in [3, §3], but not identical to it. This difference makes our construction work for non-reflexive hermitian categories and, as we shall explain in the next section, to work with systems of sesquilinear forms, where the forms can be defined with respect to different hermitian structures on $\mathscr{C}$.

3.1. **The Category of Twisted Double Arrows.** Let $(\mathscr{C}, *, \omega)$ be a hermitian category. We construct the *category of twisted double arrows in* $\mathscr{C}$, denoted $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$, as follows: The objects of $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ are quadruples (M, N, f, g) such that $f, g \in \mathrm{Hom}_{\mathscr{C}}(M, N^*)$. A morphism from (M, N, f, g) to (M', N', g', f') is a pair $(\phi, \psi^{\mathrm{op}})$ such that $\phi \in \mathrm{Hom}(M, M')$, $\psi \in \mathrm{Hom}(N', N)$, $f'\phi = \psi^* f$ and $g'\phi = \psi^* g$. The composition of two morphisms is given by $(\phi, \psi^{\mathrm{op}})(\phi', \psi'^{\mathrm{op}}) = (\phi\phi', (\psi'\psi)^{\mathrm{op}})$.

The category $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ is easily seen to be an additive category. Moreover, it has a hermitian structure: For every $(M, N, f, g) \in \widetilde{\mathrm{Ar}}_2(\mathscr{C})$, define $(M, N, f, g)^* = (N, M, g^*\omega_N, f^*\omega_N)$ and $\omega_{(M,N,f,g)} = \mathrm{id}_{(M,N,f,g)} = (\mathrm{id}_M, \mathrm{id}_N^{\mathrm{op}})$. In addition, for every morphism $(\phi, \psi^{\mathrm{op}}) : (M, N, f, g) \to (M', N', f', g')$, let $(\phi, \psi^{\mathrm{op}})^* = (\psi, \phi^{\mathrm{op}})$. It is now routine to check that $(\widetilde{\mathrm{Ar}}_2(\mathscr{C}), *, \omega)$ is a *reflexive* hermitian category. Also observe that $**$ is the just identity functor on $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$. The following proposition describes the hermitian forms over $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$.

Proposition 3.1. *Let $Z := (M, N, f, g) \in \widetilde{\mathrm{Ar}}_2(\mathscr{C})$ and let $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(M, N)$. Then $(Z, (\alpha, \beta^{\mathrm{op}}))$ is a an hermitian form over $\widetilde{\mathrm{Ar}}_2(\mathscr{C}) \iff \alpha = \beta$ and $\alpha^* f = g^*\omega_N\alpha \iff \alpha = \beta$ and $\alpha^* g = f^*\omega_N\alpha$.*

Proof. By definition, $Z^* = (N, M, g^*\omega_N, f^*\omega_N)$, so $(\alpha, \beta^{\mathrm{op}})$ is morphism from Z to Z^* if and only if $\beta^* f = g^*\omega_N\alpha$ and $\beta^* g = f^*\omega_N\alpha$. In addition, by computation, we see that $(\alpha, \beta^{\mathrm{op}}) = (\alpha, \beta^{\mathrm{op}})^* \circ \omega_Z$ precisely when $\alpha = \beta$. Therefore, $(Z, (\alpha, \beta^{\mathrm{op}}))$ is a hermitian form if and only if $\alpha = \beta$, $\alpha^* f = g^*\omega_N\alpha$ and $\alpha^* g = f^*\omega_N\alpha$. It is therefore enough to show $\alpha^* f = g^*\omega_N\alpha \iff \alpha^* g = f^*\omega_N\alpha$. Indeed, if $\alpha^* f = g^*\omega_N\alpha$, then $\alpha^*\omega_N^* g^{**} = f^*\alpha^{**}$. Therefore, $\alpha^* g = \alpha^*\omega_N^*\omega_{N^*}g = \alpha^*\omega_N^* g^{**}\omega_M = f^*\alpha^{**}\omega_M = f^*\omega_N\alpha$, as required (we used the naturality of ω and the identity $\omega_N^*\omega_{N^*} = \mathrm{id}_{N^*}$ in the computation). The other direction follows by symmetry. $\square$

Theorem 3.2. *Let $\mathscr{C}$ be a hermitian category. Define a functor $F : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ by*

$$F(M, s) = ((M, M, s^*\omega_M, s), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})),$$

$$F(\psi) = (\psi, (\psi^{-1})^{\mathrm{op}})$$

for all $(M, s) \in \mathrm{Sesq}(\mathscr{C})$ and any morphism ψ in $\mathrm{Sesq}(\mathscr{C})$. Then F induces an equivalence of categories between $\mathrm{Sesq}(\mathscr{C})$ and $\mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$.

Proof. Let $(M, s) \in \mathrm{Sesq}(\mathscr{C})$. That $F(M, s)$ does lie in $\mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ follows from Proposition 3.1. Let $\psi : (M, s) \to (M', s')$ be an isometry. Then

$$F(\psi)^*(\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}})F(\psi) = (\psi, (\psi^{-1})^{\mathrm{op}})^*(\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}})(\psi, (\psi^{-1})^{\mathrm{op}}) =$$

$$(\psi^{-1}, \psi^{\mathrm{op}})(\mathrm{id}_{M'}, \mathrm{id}_{M'}^{\mathrm{op}})(\psi, (\psi^{-1})^{\mathrm{op}}) = (\psi^{-1}\,\mathrm{id}_{M'}\,\psi, (\psi^{-1}\,\mathrm{id}_{M'}\,\psi)^{\mathrm{op}}) = (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})\ .$$

Thus, $F(\psi)$ is an isometry from $F(M, s)$ to $F(M', s')$. It is clear that F respects composition, so we conclude that F is a functor.

To see that F induces an equivalence, we construct a mutual inverse for F. Let $G : \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C})) \to \mathrm{Sesq}(\mathscr{C})$ be defined by

$$G((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) = (M, \alpha^*g)$$

$$G(\phi, \psi^{\mathrm{op}}) = \phi$$

for all $((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) \in \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ and any morphism $(\phi, \psi^{\mathrm{op}})$ in $\mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$.

Let $(Z, (\alpha, \alpha^{\mathrm{op}})), (Z', (\alpha', \alpha'^{\mathrm{op}})) \in \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ and let $(\phi, \psi^{\mathrm{op}}) : (Z, (\alpha, \alpha^{\mathrm{op}})) \to (Z', (\alpha', \alpha'^{\mathrm{op}}))$. It is easy to see that $G(Z, (\alpha, \alpha^{\mathrm{op}})) \in \mathrm{Sesq}(\mathscr{C})$, so we turn to check that $G(\phi, \psi^{\mathrm{op}})$ is an isometry from $G(Z, (\alpha, \alpha^{\mathrm{op}}))$ to $G(Z', (\alpha', \alpha'^{\mathrm{op}}))$. Writing $Z = (M, N, f, g)$ and $Z' = (M', N', f', g')$, this amounts to showing $\alpha^*g = \phi^*\alpha'^*g'\phi$. Indeed, since $(\phi, \psi^{\mathrm{op}})$ is morphism from Z to Z', we have $g'\phi = \psi^*g$, and since $(\phi, \psi^{\mathrm{op}})$ is an isometry, we also have $(\phi, \psi^{\mathrm{op}})^*(\alpha', \alpha'^{\mathrm{op}})(\phi, \psi^{\mathrm{op}}) = (\alpha, \alpha^{\mathrm{op}})$, which in turn implies $\psi\alpha'\phi = \alpha$. We now have $\phi^*\alpha'^*g'\phi = \phi^*\alpha'^*\psi^*g = (\psi\alpha'\phi)^*g = \alpha^*g$, as required. That G preserves composition is straightforward.

It is easy to see that GF is the identity functor on $\mathrm{Sesq}(\mathscr{C})$, so it is left to show that there is a natural isomorphism from FG to $\mathrm{id}_{\mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))}$. Keeping the notation of the previous paragraph, we have

$$FG((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) = ((M, M, (\alpha^*g)^*\omega_M, \alpha^*g), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}))\ .$$

By Proposition 3.1, we have $\alpha^*f = g^*\omega_N\alpha$, hence $(\alpha^*g)^*\omega_M = g^*\alpha^{**}\omega_M = g^*\omega_N\alpha = \alpha^*f$. Thus,

$$(2) \qquad FG((M, N, f, g), (\alpha, \alpha^{\mathrm{op}})) = ((M, M, \alpha^*f, \alpha^*g), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}}))\ .$$

Define a natural isomorphism $t : \mathrm{id}_{\mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))} \to FG$ by $t_{(Z, (\alpha, \alpha^{\mathrm{op}}))} = (\mathrm{id}_M, \alpha^{\mathrm{op}})$. Using (2), it is easy to see that $t_{(Z, (\alpha, \alpha^{\mathrm{op}}))}$ is indeed an isometry from $(Z, (\alpha, \alpha^{\mathrm{op}}))$ to $FG(Z, (\alpha, \alpha^{\mathrm{op}}))$. The map t is natural since for $Z', (\phi, \psi^{\mathrm{op}})$ as above, we have $FG(\phi, \psi^{\mathrm{op}})t_{(Z, (\alpha, \alpha^{\mathrm{op}}))} = (\phi, (\phi^{-1})^{\mathrm{op}})(\mathrm{id}_M, \alpha^{\mathrm{op}}) = (\phi, (\alpha\phi^{-1})^{\mathrm{op}}) = (\phi, (\psi\alpha')^{\mathrm{op}}) = (\mathrm{id}_{M'}, \alpha'^{\mathrm{op}})(\phi, \psi^{\mathrm{op}}) = t_{(Z', (\alpha', \alpha'^{\mathrm{op}}))}(\phi, \psi^{\mathrm{op}})$ (we used the identity $\psi\alpha'\phi = \alpha$ verified above). $\square$

Remark 3.3. On the model of [3, §3], one can also construct the *category of (non-twisted) double arrows in* $\mathscr{C}$, denoted $\mathrm{Ar}_2(\mathscr{C})$. Its objects are quadruples (M, N, f, g) with $M, N \in \mathscr{C}$ and $f, g \in \mathrm{Hom}(M, N)$. A morphism from (M, N, f, g) to (M', N', f', g') is a pair (ϕ, ψ), where $\phi \in \mathrm{Hom}(M, M')$ and $\psi \in \mathrm{Hom}(N, N')$ satisfy $\psi f = f'\phi$ and $\psi g = g'\phi$. The category $\mathrm{Ar}_2(\mathscr{C})$ is obviously

additive and moreover, it admits a hermitian structure given by $(M, N, f, g)^* = (N^*, M^*, g^*, f^*)$, $(\phi, \psi)^* = (\psi^*, \phi^*)$ and $\omega_{(M,N,f,g)} = (\omega_M, \omega_N)$.

There is a functor $T : \widetilde{\mathrm{Ar}}_2(\mathscr{C}) \to \mathrm{Ar}_2(\mathscr{C})$ given by $T(M, N, f, g) = (M, N^*, f, g)$ and $T(\phi, \psi^{\mathrm{op}}) = (\phi, \psi^*)$. This functor induces an equivalence if $\mathscr{C}$ is reflexive, but otherwise it need neither be faithful nor full. In addition, provided $\mathscr{C}$ is reflexive, one can define a functor $F' : \mathrm{Sesq}(\mathscr{C}) \to \mathrm{UH}(\mathrm{Ar}_2(\mathscr{C}))$ by $F'(M, s) = ((M, M^*, s^*\omega_M, s), (\omega_M, \mathrm{id}_{M^*}))$ and $F'(\psi) = (\psi, (\psi^{-1})^*)$. This functor induces an equivalence of categories; the proof is analogues to [3, Th. 4.1].

3.2. **Hyperbolic Sesquilinear Forms.** Let $\mathscr{C}$ be a hermitian category. The equivalence $\mathrm{Sesq}(\mathscr{C}) \sim \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ of Theorem 3.2 allows us to pull back notions defined for unimodular hermitian forms over $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ to sesquilinear form over $\mathscr{C}$. In this subsection, we will do this for hyperbolicity and thus obtain a notion of a Witt group of sesquilinear forms.

Throughout, F denotes the functor $\mathrm{Sesq}(\mathscr{C}) \to \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ defined in Theorem 3.2.

Definition 3.4. *A sesquilinear form (M, s) over $\mathscr{C}$ is called* hyperbolic *if $F(M, s)$ is hyperbolic as unimodular hermitian form over $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$.*

The following proposition gives a more concrete meaning to hyperbolicity of sesquilinear forms over $\mathscr{C}$.

Proposition 3.5. *Up to isometry, the hyperbolic sesquilinear forms over $\mathscr{C}$ are given by*

$$(M \oplus N, \begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix})$$

where $M, N \in \mathscr{C}$, $f \in \mathrm{Hom}_{\mathscr{C}}(N, M^*)$, $g \in \mathrm{Hom}_{\mathscr{C}}(M, N^*)$ and $\begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix}$ is an element of $\mathrm{Hom}_{\mathscr{C}}(M \oplus N, M^* \oplus N^*)$ given in matrix form. Furthermore, a unimodular ϵ-hermitian form is hyperbolic as a sesquilinear form (i.e. in the sense of Definition 3.4) if and only if it is hyperbolic as a unimodular ϵ-hermitian form (see section 2).

Proof. Let G be the functor $\mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C})) \to \mathrm{Sesq}(\mathscr{C})$ defined in the proof of Theorem 3.2. Since G is a mutual inverse of F, the hyperbolic sesquilinear forms over $\mathscr{C}$ are the forms isometric to $G(Z \oplus Z^*, \mathbb{H}_Z)$ for $Z \in \widetilde{\mathrm{Ar}}_2(\mathscr{C})$. Write $Z = (M, N, h, g)$. Then

$$(Z \oplus Z^*, \mathbb{H}_Z) = \left((M \oplus N, N \oplus M, \begin{bmatrix} h & 0 \\ 0 & g^*\omega_N \end{bmatrix}, \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix}), \begin{bmatrix} 0 & \mathrm{id}_{Z^*} \\ \omega_Z & 0 \end{bmatrix}\right).$$

Observe that $\begin{bmatrix} 0 & \mathrm{id}_{Z^*} \\ \omega_Z & 0 \end{bmatrix} = \begin{bmatrix} 0 & (\mathrm{id}_N, \mathrm{id}_M^{\mathrm{op}}) \\ (\mathrm{id}_M, \mathrm{id}_N^{\mathrm{op}}) & 0 \end{bmatrix} = (\begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}^{\mathrm{op}})$. Thus,

$$G(Z \oplus Z^*, \mathbb{H}_Z) = (M \oplus N, \begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}^* \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix}),$$

and since $\begin{bmatrix} 0 & \mathrm{id}_N \\ \mathrm{id}_M & 0 \end{bmatrix}^* \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{id}_{M^*} \\ \mathrm{id}_{N^*} & 0 \end{bmatrix} \begin{bmatrix} g & 0 \\ 0 & h^*\omega_N \end{bmatrix} = \begin{bmatrix} 0 & h^*\omega_N \\ g & 0 \end{bmatrix}$, we see that $G(Z \oplus Z^*, \mathbb{H}_Z)$ matches to the description of the proposition. Furthermore, by

putting $h = f^* \omega_M$ for $f \in \mathrm{Hom}_{\mathscr{C}}(N, M^*)$, we get $h^* \omega_N = \omega_M^* f^{**} \omega_N = \omega_M^* \omega_{M^*} f = f$. Thus, $(M \oplus N, \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right])$ is hyperbolic for all M, N, f, g as in the proposition.

To finish, note that we have clearly shown that $(Q \oplus Q^*, \mathbb{H}_Q^\epsilon)$ is hyperbolic as a sesquilinear form for every $Q \in \mathscr{C}$. To see the converse, assume $(M \oplus N, \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right])$ is ϵ-hermitian and unimodular. Then

$$\left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right] = \epsilon \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]^* \omega_{M \oplus N} = \epsilon \left[\begin{smallmatrix} 0 & g^* \\ f^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \omega_M & 0 \\ 0 & \omega_N \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & \epsilon g^* \omega_N \\ \epsilon f^* \omega_M & 0 \end{smallmatrix}\right],$$

hence $g = \epsilon f^* \omega_N$ and $f = \epsilon g^* \omega_M$. Since $\left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right]$ is unimodular, f and g are bijective and hence, so are ω_N and ω_M. In particular, M is reflexive. It is now routine to verify that the map $\mathrm{id}_M \oplus f : M \oplus N \to M \oplus M^*$ is an isometry from $(M \oplus N, \left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right])$ to $(M \oplus M^*, \mathbb{H}_M^\epsilon)$, so the former is hyperbolic in the sense of section 2. $\qquad\square$

Let (A, σ) be a ring with involution. In case $\mathscr{C}$ is the category of right A-modules, considered as a hermitian category as in Example 2.1, we obtain a notion of hyperbolic sesquilinear forms over (A, σ). These hyperbolic forms can be characterized as follows.

Proposition 3.6. *A sesquilinear form (M, s) over (A, σ) is hyperbolic if and only if there are submodules $M_1, M_2 \le M$ such that $s(M_1, M_1) = s(M_2, M_2) = 0$ and $M = M_1 \oplus M_2$. Furthermore, if (M, s) is unimodular and ϵ-hermitian, then (M, s) is hyperbolic as a sesquilinear space if and only if it is hyperbolic as an ϵ-hermitian unimodular space.*

Proof. Recall that for any two right A-modules M_1, M_2, we identify $(M_1 \oplus M_2)^*$ with $M_1^* \oplus M_2^*$ via $f \leftrightarrow (f|_{M_1}, f|_{M_2})$. Let (M, s) be a sesquilinear space and assume $M = M_1 \oplus M_2$. By straightforward computation, we see that s_r is of the form $\left[\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right] \in \mathrm{Hom}_A(M, M^*) = \mathrm{Hom}_A(M_1 \oplus M_2, M_1^* \oplus M_2^*)$ if and only if $s(M_1, M_1) = s(M_2, M_2) = 0$. The proposition therefore follows from Proposition 3.5. $\qquad\square$

Returning to the general case, let $\mathrm{WG}_S(\mathscr{C})$ be the Grothendieck group of isometry classes of *sesquilinear* forms over $\mathscr{C}$, with respect to orthogonal sum. It is easy to see that the hyperbolic isometry classes span a subgroup of $\mathrm{WG}_S(\mathscr{C})$, which we denote by $\mathbb{H}(\mathscr{C})$. The *Witt group of sesquilinear forms* over $\mathscr{C}$ is defined to the quotient

$$\mathrm{W}_S(\mathscr{C}) = \mathrm{WG}_S(\mathscr{C})/\mathbb{H}(\mathscr{C}) \ .$$

By definition, we have $\mathrm{W}_S(\mathscr{C}) \cong \mathrm{W}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$. Taking $\mathscr{C}$ to be the category of all (resp. reflexive, projective) right A-modules of finite type and their duals,[2] we obtain a notion of a Witt group for sesquilinear forms over (A, σ). Also observe that there is a homomorphism of groups $\mathrm{W}^\epsilon(\mathscr{C}) \to \mathrm{W}_S(\mathscr{C})$ given by sending the class of a unimodular ϵ-hermitian form to its corresponding class in

---

[2] It is possible that M^* would not be of finite type when M is. However, when A is noetherian, M^* is of finite type because any surjection $A^n \twoheadrightarrow M$ gives rise to an injection $M^* \hookrightarrow (A^*)^n$.

$W_S(\mathscr{C})$. Corollary 5.11 below presents sufficient conditions for the injectivity of this homomorphism.

3.3. **Extension of Scalars.** Let R/K be a commutative ring extension and let $\mathscr{C}$ be a K-linear hermitian category. Then the category $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ is also K-linear. For later usage, we now check that the scalar extension functor $\mathcal{R}_{R/K}$ of subsection 2.4 "commutes" with the functor F of Theorem 3.2.

Proposition 3.7. *There is a 1-hermitian duality preserving functor $J : \widetilde{\mathrm{Ar}}_2(\mathscr{C})_R \to \widetilde{\mathrm{Ar}}_2(\mathscr{C}_R)$ given by*

$$J((M, N, f, g)_R) = (M_R, N_R, f_R, g_R),$$
$$J((\phi, \psi^{\mathrm{op}}) \otimes a) = (\phi \otimes a, (\psi \otimes a)^{\mathrm{op}}),$$

for all $(M, N, f, g) \in \widetilde{\mathrm{Ar}}_2(\mathscr{C})$ and any morphism $(\phi, \psi^{\mathrm{op}})$ in $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$. (The associated natural isomorphism $i : J* \to *J$ is the identity map.) The functor J is faithful and full, and it makes the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{Sesq}(\mathscr{C}) & \xrightarrow{\quad\quad F \quad\quad} & \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C})) \\
\downarrow{\scriptstyle \mathcal{R}_{R/K}} & & \downarrow{\scriptstyle \mathcal{R}_{R/K}} \\
\mathrm{Sesq}(\mathscr{C}_R) \xrightarrow{F} \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}_R)) & \xleftarrow{\;\;J\;\;} & \mathrm{UH}(\widetilde{\mathrm{Ar}}_2(\mathscr{C})_R)
\end{array}
$$

Proof. We only check that J is faithful and full. All other assertions follow by computation. Let $Z, Z' \in \widetilde{\mathrm{Ar}}_2(\mathscr{C})$. Define $I : \mathrm{Hom}_{\widetilde{\mathrm{Ar}}_2(\mathscr{C}_R)}(JZ_R, JZ'_R) \to \mathrm{Hom}_{\widetilde{\mathrm{Ar}}_2(\mathscr{C})_R}(Z_R, Z'_R)$ by

$$I\Big(\sum_i f_i \otimes a_i, \big(\sum_j g_j \otimes b_j\big)^{\mathrm{op}}\Big) = \sum_{i,j} \Big((f_i, 0^{\mathrm{op}}) \otimes a_i + (0, g_j^{\mathrm{op}}) \otimes b_j \Big) \ .$$

Then it is routine to verify that I is an inverse of $J : \mathrm{Hom}_{\widetilde{\mathrm{Ar}}_2(\mathscr{C})_R}(Z_R, Z'_R) \to \mathrm{Hom}_{\widetilde{\mathrm{Ar}}_2(\mathscr{C}_R)}(JZ_R, JZ'_R)$. Thus, J is full and faithful. $\qquad\square$

As an immediate corollary, we get:

Corollary 3.8. *Let (M, s), (M', s') be two sesquilinear forms over $\mathscr{C}$. Then $\mathcal{R}_{R/K}(M, s)$ is isometric to $\mathcal{R}_{R/K}(M', s')$ if and only if $\mathcal{R}_{R/K}F(M, s)$ is isometric to $\mathcal{R}_{R/K}F(M', s')$.*

4. Systems of Sesquilinear Forms

In this section, we explain how to generalize the results of Section 3 to systems of sesquilinear forms.

Let A be a ring and let $\{\sigma_i\}_{i \in I}$ be an nonempty family of (not necessarily distinct) involutions of A. A system of sesquilinear forms over $(A, \{\sigma_i\}_{i \in I})$ is a pair $(M, \{s_i\}_{i \in I})$ such that (M, s_i) is a sesquilinear space over (A, σ_i) for all i. An isometry between two systems of sesquilinear forms $(M, \{s_i\}_{i \in I})$, $(M', \{s'_i\}_{i \in I})$ is an isomorphism $f : M \to M'$ such that $s'_i(fx, fy) = s_i(x, y)$ for all $x, y \in M$.

Observe that each of the involutions σ_i gives rise to a hermitian structure $(*_i, \omega_i)$ on Mod-A, the category of right A-modules. In particular, a system of sesquilinear forms $(M, \{s_i\})$ gives rise to homomorphisms $(s_i)_r, (s_i)_\ell : M \to M^{*i}$ given by $(s_i)_r(x)(y) = \sigma_i(s_i(y, x))$ and $(s_i)_\ell(x)(y) = s_i(x, y)$, where $M^{*i} = \mathrm{Hom}_A(M, A)$, considered as a right A-module via the action $(f \cdot a)m = \sigma_i(a)f(m)$. This leads to the notion of systems of sesquilinear forms over hermitian categories.

Let $\mathscr{C}$ be an additive category and let $\{*_i, \omega_i\}_{i \in I}$ be a nonempty family of hermitian structures on $\mathscr{C}$. A system of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\}_{i \in I})$ is a pair $(M, \{s_i\}_{i \in I})$ such that $M \in \mathscr{C}$ and (M, s_i) is a sesquilinear form over $(\mathscr{C}, *_i, \omega_i)$. An isometry between two systems of sesquilinear forms $(M, \{s_i\}_{i \in I})$ and $(M', \{s'_i\}_{i \in I})$ is an isomorphism $f : M \xrightarrow{\sim} M'$ such that $f^{*i} s'_i f = s_i$ for all $i \in I$. We let $\mathrm{Sesq}_I(\mathscr{C})$ (or $\mathrm{Sesq}_I(\mathscr{C}, \{*_i, \omega_i\})$) denote the category of systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\}_{i \in I})$ with isometries as morphisms.

Keeping the notation of the previous paragraph, the results of section 3 can be extended to systems of sesquilinear forms as follows: Define the category of *twisted double I-arrows* over $(\mathscr{C}, \{*_i, \omega_i\}_{i \in I})$, denoted $\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})$, to be the category whose objects are quadruples $(M, N, \{f_i\}_{i \in I}, \{g_i\}_{i \in I})$ with $M, N \in \mathscr{C}$ and $f_i, g_i \in \mathrm{Hom}_\mathscr{C}(M, N^{*i})$. A morphism $(M, N, \{f_i\}, \{g_i\}) \to (M', N', \{f'_i\}, \{g'_i\})$ is a formal pair $(\phi, \psi^{\mathrm{op}})$ such that $\phi \in \mathrm{Hom}(M, M')$, $\psi \in \mathrm{Hom}(N', N)$ and $\psi^{*i} f_i = f'_i \phi$, $\psi^{*i} g_i = g'_i \phi$ for all $i \in I$. The composition is defined by the formula $(\phi, \psi^{\mathrm{op}})(\phi', \psi'^{\mathrm{op}}) = (\phi \phi', (\psi' \psi)^{\mathrm{op}})$.

The category $\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})$ can be made into a reflexive hermitian category by letting $(M, N, \{f_i\}, \{g_i\})^* = (N, M, \{g_i^{*i} \omega_{i,N}\}, \{f_i^{*i} \omega_{i,M}\})$, $(\phi, \psi^{\mathrm{op}})^* = (\psi, \phi^{\mathrm{op}})$ and $\omega_{(M, N, \{f_i\}, \{g_i\})} = (\mathrm{id}_M, \mathrm{id}_N^{\mathrm{op}})$. It is now possible to prove the following theorem, whose proof is completely analogous to the proof of Theorem 3.2.

Theorem 4.1. *Define a functor $F : \mathrm{Sesq}_I(\mathscr{C}) \to \mathrm{UH}(\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C}))$ by*

$$F(M, \{s_i\}) = ((M, M, \{s_i^{*i} \omega_{i,M}\}, \{s_i\}), (\mathrm{id}_M, \mathrm{id}_M^{\mathrm{op}})),$$

$$F(\psi) = (\psi, (\psi^{-1})^{\mathrm{op}})$$

Then F induces an equivalence of categories.

Proof (sketch). It is easy to see that that any hermitian form over $\mathrm{UH}(\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C}))$ has the form $((M, N, \{f_i\}, \{g_i\}), (\alpha, \alpha^{\mathrm{op}}))$. Define a functor $G : \mathrm{UH}(\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})) \to \mathrm{Sesq}_I(\mathscr{C})$ by

$$G((M, N, \{f_i\}, \{g_i\}), (\alpha, \alpha^{\mathrm{op}})) = (M, \{\alpha^{*i} g_i\}),$$

$$G(\phi, \psi^{\mathrm{op}}) = \phi \ .$$

By arguing as in the proof of Theorem 3.2, we see that G is a mutual inverse of F. $\qquad \square$

As we did in section 3, we can use Theorem 4.1 to define hyperbolic systems of sesquilinear forms. Namely, a system of forms $(M, \{s_i\})$ over $\mathscr{C}$ will be called *hyperbolic* if $F(M, \{s_i\})$ is hyperbolic over $\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})$. The following two propositions are proved in the same manner as Propositions 3.5 and 3.6, respectively.

Proposition 4.2. *A system of sesquilinear forms $(M, \{s_i\})$ over $\mathscr{C}$ is hyperbolic if and only if there are $M_1, M_2 \in \mathscr{C}$, $f_i \in \mathrm{Hom}(M_2, M_1^{*_i})$, $g_i \in \mathrm{Hom}(M_1, M_2^{*_i})$ such that $M = M_1 \oplus M_2$ and for all $i \in I$,*

$$s_i = \begin{bmatrix} 0 & f_i \\ g_i & 0 \end{bmatrix} \in \mathrm{Hom}(M, M^{*_i}) = \mathrm{Hom}(M_1 \oplus M_2, M_1^{*_i} \oplus M_2^{*_i}) \ .$$

In this case, each of the sesquilinear forms (M, s_i) (over $(\mathscr{C}, *_i, \omega_i)$) is hyperbolic.

Proposition 4.3. *Let A be a ring and let $\{\sigma_i\}_{i \in I}$ be a nonempty family of involutions of A. A system of sesquilinear forms $(M, \{s_i\})$ over $(A, \{\sigma_i\})$ is hyperbolic if and only if there are submodules $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$ and $s_i(M_1, M_1) = s_i(M_2, M_2) = 0$ for all $i \in I$. In this case, each of the sesquilinear forms (M, s_i) (over (A, σ_i)) is hyperbolic.*

The notion of hyperbolic systems of sesquilinear forms can be used to define Witt groups. We leave the details to the reader.

Let R/K be a commutative ring extension. If $\mathscr{C}$ and all the hermitian structures $\{*_i, \omega_i\}_{i \in I}$ are K-linear, then the scalar extension functor $\mathcal{R}_{R/K} : \mathscr{C} \to \mathscr{C}_R$ is 1-hermitian and duality preserving with respect to $(*_i, \omega_i)$ for all $i \in I$. Therefore, we have a functor $\mathcal{R}_{R/K} : \mathrm{Sesq}_I(\mathscr{C}) \to \mathrm{Sesq}_I(\mathscr{C}_R)$ given by $\mathcal{R}_{R/K}(M, \{s_i\}_{i \in I}) = (M_R, \{(s_i)_R\}_{i \in I})$. We thus have a notion of scalar extension for systems of bilinear forms (and it agrees with the obvious scalar extension for systems of bilinear forms over a ring with a family of involutions, provided the assumptions of Remark 2.2). Using the ideas of subsection 3.3, one can show:

Corollary 4.4. *Let $(M, \{s_i\})$, $(M', \{s_i'\})$ be two systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$. Then $\mathcal{R}_{R/K}(M, \{s_i\})$ is isometric to $\mathcal{R}_{R/K}(M', \{s_i'\})$ if and only if $\mathcal{R}_{R/K}F(M, \{s_i\})$ is isometric to $\mathcal{R}_{R/K}F(M', \{s_i'\})$.*

5. Applications

The following section uses the previous results to generalize various known results about hermitian forms (over rings or reflexive hermitian categories) to systems of sesquilinear forms over (not-necessarily reflexive) hermitian categories. Some of the consequences to follow were obtained in [3] for hermitian forms over rings. Here we rephrase them for hermitian categories, extend them to systems of sesquilinear forms and drop the assumption that the base module (or object) is reflexive.

5.1. **Witt's Cancelation Theorem.** Quebbemann, Scharlau and Schulte have proven Witt's Cancelation Theorem for unimodular hermitian forms over hermitian categories satisfying certain assumptions ([11, §3.4]). It is possible use their result with Theorem 4.1 to get a cancelation theorem for systems of sesquilinear forms (in the same manner as in [3, §6]). However, as pointed out in [11], there is an even stronger cancelation theorem due to Reiter ([12, Th. 6.2]), that applies to hermitian forms over semilocal rings.[3] (Reiter actually proved a version of Witt's Extension Theorem; the cancelation follows as a corollary.)

Theorem 5.1 (Reiter). *Let (R, σ) be a semilocal ring with involution and let $(M, s), (M', s'), (M'', s'')$ be unimodular 1-hermitian (resp. (-1)-hermitian) forms over (R, σ). Assume that $\frac{1}{2} \in R$ and M, M', M'' are projective of finite type. Then $(M, s) \oplus (M', s') \simeq (M, s) \oplus (M'', s'') \iff (M', s') \simeq (M'', s'').$*

Proof. Let (N, t) be another copy of (M, s) and identify $(M \oplus M', s \oplus s')$ with $(N \oplus M'', t \oplus s'')$. It is enough to prove that the isometry $(M, s) \simeq (N, t)$ extends to an isometry of $(M \oplus M', s \oplus s')$ (as the extension would send $M' = M^{\perp}$ to $M'' = N^{\perp}$). But this follows from [12, Th. 6.2].   □

In order to apply Reiter's result, we recall that if R is a ring and $e \in R$ is an idempotent, then R is semilocal if and only if eRe and $(1 - e)R(1 - e)$ are semilocal. This implies that if M, N are two objects in an additive category, then $\operatorname{End}(M \oplus N)$ is semilocal if and only if $\operatorname{End}(M)$ and $\operatorname{End}(N)$ are semilocal. We will also make use of a result by Camps and Dicks ([6, Cor. 2]):

Theorem 5.2 (Camps, Dicks). *Let R be a semilocal ring and let S be a subring of R such that $S^{\times} = S \cap R^{\times}$ (i.e. every element of S that is invertible in R is also invertible in S). Then S is semilocal.*

Theorem 5.3 (Cancelation Theorem). *Let $\mathscr{C}$ be an additive category admitting hermitian structures $\{*_i, \omega_i\}_{i \in I}$ $(I \neq \phi)$, and let $(M, \{s_i\}), (M', \{s'_i\}), (M'', \{s''_i\})$ be systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$. Assume that $\operatorname{End}(M), \operatorname{End}(M'), \operatorname{End}(M'')$ are semilocal rings in which 2 is invertible. Then*

$$(M, \{s_i\}) \oplus (M', \{s'_i\}) \simeq (M, \{s_i\}) \oplus (M'', \{s''_i\}) \iff (M', \{s'_i\}) \simeq (M'', \{s''_i\}).$$

Proof. By Theorem 4.1, it is enough to verify $F(M', \{s'_i\}) \simeq F(M'', \{s''_i\})$. Write $(Z, (\alpha, \alpha^{\mathrm{op}})) = F(M, \{s_i\}) \oplus F(M', \{s'_i\})$ and $E = \operatorname{End}_{\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})}(Z)$. By applying transfer with respect to $(Z, (\alpha, \alpha^{\mathrm{op}}))$ (i.e. the functor $\mathrm{T}_{(Z, (\alpha, \alpha^{\mathrm{op}}))}$ of subsection 2.3), we reduce to showing cancelation for unimodular 1-hermitian forms defined on projective E-modules of finite type. It is easy to see that $\frac{1}{2} \in E$, so we are done by Theorem 5.1 if we prove that E is semilocal. Write $(W, (\beta, \beta^{\mathrm{op}})) = F(M, \{s_i\})$ and $(W', (\beta', \beta'^{\mathrm{op}})) = F(M_i, \{s'_i\})$. By the previous comments, it is enough to show $\operatorname{End}(W)$ and $\operatorname{End}(W')$ are semilocal. Indeed, $\operatorname{End}(W)$ is a

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[3] Recall that a ring R is semilocal if $R/\operatorname{Jac}(R)$ is semisimple artinian, where $\operatorname{Jac}(R)$ is the Jacobson radical of R.

subring of $\operatorname{End}(M) \times \operatorname{End}(M)^{\mathrm{op}}$ and it is easy to check that any endomorphism $(\phi, \psi^{\mathrm{op}}) \in \operatorname{End}(W)$ which is invertible in $\operatorname{End}(M) \times \operatorname{End}(M)^{\mathrm{op}}$ is also invertible in $\operatorname{End}(W)$. Thus, by Theorem 5.2, $\operatorname{End}(W)$ is semilocal, and in the same manner, so is $\operatorname{End}(W')$. $\square$

As a special case of the theorem, we get:

Corollary 5.4. *Let A be a ring with $\frac{1}{2} \in A$ and let $\{\sigma_i\}_{i \in I}$ be a nonempty family of involutions on A. Let $(M, \{s_i\})$, $(M', \{s_i'\})$, $(M'', \{s_i''\})$ be systems of sesquilinear forms over $(A, \{\sigma_i\})$. Assume that M, M', M'' are artinian or A is semilocal and M, M', M'' are finitely presented. Then*

$$(M, \{s_i\}) \oplus (M', \{s_i'\}) \simeq (M, \{s_i\}) \oplus (M'', \{s_i''\}) \iff (M', \{s_i'\}) \simeq (M'', \{s_i''\}).$$

Proof. If M, M', M'' are artinian, then their endomorphism rings are semilocal by [6, Cor. 6]. If A is semilocal and M, M', M'' are finitely presented, then their endomorphism rings are semilocal by [7, Th. 3.3]. Now apply Theorem 5.3. $\square$

5.2. **Finiteness Results.** In the following two subsections, we generalize the finiteness results of [3, §10] to systems of sesquilinear forms.

For a ring A, we denote by $T(A)$ the $\mathbb{Z}$-torsion subgroup of A. Recall that if R is a commutative ring, A is said to be *R-finite* if $A_R = A \otimes_{\mathbb{Z}} R$ is a finitely generated R-module and $T(A)$ is finite. Note that being R-finite passes to subrings.

The proofs of the results to follow are completely analogous to the proofs of the corresponding statements in [3, §9]; they are based on applying the equivalence of Theorem 4.1 and then use the finiteness results of [1], possibly after applying transfer.

Throughout, $\mathscr{C}$ is an additive category and $\{*_i, \omega_i\}_{i \in I}$ is a nonempty family of hermitian structures on $\mathscr{C}$. Fix a system of sesquilinear forms $(V, \{s_i\}_{i \in I})$ over $(\mathscr{C}, \{_i, \omega_i\})$ and let $Z(V, \{s_i\}) = (V, V, \{s_i^{*_i}\omega_{i,V}\}, \{s_{ir}\}) \in \widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})$. (Note that $F(V, \{s_i\}) = (Z, (\mathrm{id}_V, \mathrm{id}_V^{\mathrm{op}}))$ with F as in Theorem 4.1).

Theorem 5.5. *If there exists a non-zero integer m such that $\operatorname{End}_{\mathscr{C}}(V)$ is $\mathbb{Z}[1/m]$-finite, then there are finitely many isometry classes of summands of $(V, \{s_i\})$.*

Theorem 5.6. *Assume that there is a non-zero integer m such that the ring $\operatorname{End}_{\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})}(Z(V, \{s_i\}))$ is $\mathbb{Z}[1/m]$-finite (e.g. if $\operatorname{End}_{\mathscr{C}}(V)$ is $\mathbb{Z}[1/m]$-finite). Then there exist only finitely many isometry classes of systems of sesquilinear forms $(V', \{s_i'\}_{i \in I})$ over $\mathscr{C}$ such that $Z(V', \{s_i'\}) \simeq Z(V, \{s_i\})$ (as objects in $\widetilde{\mathrm{Ar}}_{2I}(\mathscr{C})$).*

5.3. **Finiteness of The Genus.** Let $\mathscr{C}$ be a hermitian category admitting a nonempty family of hermitian structures $\{*_i, \omega_i\}_{i \in I}$. We say that two systems of sesquilinear forms $(M, \{s_i\})$, $(M', \{s_i'\})$ are *of the same genus* if they become isometric after applying $\mathcal{R}_{\mathbb{Z}_p/\mathbb{Z}}$ for every prime number p (where $\mathbb{Z}_p$ are the p-adic integer). (See Remark 2.2 for conditions under which this definition of genus agrees with the naive definition of genus for module categories.) As in [3, Th. 10.3], we have:

Theorem 5.7. *Let $(M, \{s_i\})$ be a system of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$, and assume that $\mathrm{End}(M)$ is $\mathbb{Q}$-finite. Then the genus of $(M, \{s_i\})$ contains only a finite number of isometry classes of systems of sesquilinear forms.*

5.4. **Forms That Are Trivial in The Witt Group.** Let $\mathscr{C}$ be a hermitian category. By definition, a unimodular ϵ-hermitian (resp. sesquilinear) form (M, s) is trivial in in $\mathrm{W}^\epsilon(\mathscr{C})$ (resp. $\mathrm{W}_S(\mathscr{C})$) if and only if there are unimodular ϵ-hermitian (resp. sesquilinear) hyperbolic forms (H_1, h_1), (H_2, h_2) such that $(M, s) \oplus (H_1, h_1) \simeq (H_2, h_2)$. In this section, we will show that under mild assumptions, this implies that (M, s) is hyperbolic.

Lemma 5.8. *Let $M \in \mathscr{C}$, and assume that M is a (finite) direct sum of objects with local endomorphism ring. Then, up to isometry, there is at most one ϵ-hermitian hyperbolic form on M.*

Proof. For $X \in \mathscr{C}$, let $[X]$ denote the isomorphism class of X. The Krull-Schmidt Theorem (e.g. see [15, pp. 237]) implies that if $M \cong \bigoplus_{i=1}^t M_i$ with each M_i indecomposable, then the unordered list $[M_1], \dots, [M_t]$ is determined by M.

Let (M, s) be an ϵ-hermitian hyperbolic form on M, say $(M, s) \simeq (N \oplus N^*, \mathbb{H}_N^\epsilon)$. Write $N \cong \oplus_{i=1}^r N_i$ with each N_i indecomposable. Then $s \simeq \bigoplus_{i=1}^r \mathbb{H}_{N_i}^\epsilon$. It is easy to check that isometry class of $\mathbb{H}_{N_i}^\epsilon$ depends only on the set $\{[N_i], [N_i^*]\}$. Furthermore, using the Krull-Schmidt Theorem, one easily verifies that the unordered list $\{[N_1], [N_1^*]\}, \dots, \{[N_r], [N_r^*]\}$ is uniquely determined by M. It follows that (M, s) is isometric to a sesquilinear form which is determined by M up to isometry. $\square$

Proposition 5.9. *Let $\mathscr{C}$ be a hermitian category such that every object in $\mathscr{C}$ is a sum of objects with local endomorphism ring in which 2 is invertible. Then a unimodular ϵ-hermitian form (M, s) is trivial in $\mathrm{W}^\epsilon(\mathscr{C})$ if and only if it is hyperbolic.*

Proof. Let (M, s) be a unimodular ϵ-hermitian form such that $(M, s) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C})$. Then there are unimodular ϵ-hermitian hyperbolic forms (H_1, h_1), (H_2, h_2) such that $(M, s) \oplus (H_1, h_1) \simeq (H_2, h_2)$. Using the Krull-Schmidt Theorem, it is easy to see that there is $N \in \mathscr{C}$ such that $M \cong N \oplus N^*$. Thus, we may consider $\mathbb{H}_N^\epsilon$ as a hermitian form on M. By Lemma 5.8, we have $\mathbb{H}_N^\epsilon \oplus h_1 \simeq h_2$, implying $\mathbb{H}_N^\epsilon \oplus h_2 \simeq s \oplus h_2$. Therefore, by Theorem 5.3, $s \simeq \mathbb{H}_N^\epsilon$, as required. $\square$

Proposition 5.10. *Let K be a field of characteristic not 2 and let $\mathscr{C}$ be a K-linear hermitian category such that all idempotents in $\mathscr{C}$ split and $\dim_K \mathrm{Hom}(M, N) < \infty$ for all $M, N \in \mathscr{C}$. Then a sesquilinear form (M, s) is trivial in $\mathrm{W}_S(\mathscr{C})$ if and only if it is hyperbolic.*

Proof. It is enough to verify that $F(M, s)$ is hyperbolic in $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ (Theorem 3.2). Our assumptions now imply that the Hom-sets in $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ are finite dimensional K-vector spaces (since they are subalgebras of $\mathrm{Hom}(M) \times \mathrm{Hom}(N)^{\mathrm{op}}$ with $M, N \in \mathscr{C}$). Therefore, every object of $\widetilde{\mathrm{Ar}}_2(\mathscr{C})$ is a sum of objects with a local endomorphism ring (e.g. see [16, §7.11]), so we are done by Proposition 5.9. $\square$

Corollary 5.11. *Under the assumptions of Proposition 5.10, the map* $\mathrm{W}(\mathscr{C}) \to \mathrm{W}_S(\mathscr{C})$ *is injective.*

Proof. This follows from Propositions 5.10 and 3.5. $\qquad\square$

5.5. **Odd Degree Extensions.** Throughout this subsection, L/K is an odd degree field extension and char $K \neq 2$. A well known theorem of Springer asserts that two unimodular hermitian forms over K become isometric over L if and only if they are already isometric over K. Moreover, the restriction map (i.e. the scalar extension map) $r_{L/K} : \mathrm{W}(K) \to \mathrm{W}(L)$ is injective. Both statements were extended to hermitian forms over f.d. K-algebras in [2, Pr. 1.2 and Th. 2.1] (see also [8] for a version in which L/K is replaced with an extension of complete discrete valuation rings). In this section, we extend these results to sesquilinear forms over hermitian categories.

Theorem 5.12. *Let $\mathscr{C}$ be an additive K-category such that $\dim_K \mathrm{Hom}(M, M')$ is finite for all $M, M' \in \mathscr{C}$. Let $\{*_i, \omega_i\}_{i \in I}$ be a nonempty family of K-linear hermitian structures on $\mathscr{C}$ and let $(M, \{s_i\})$, $(M', \{s_i'\})$ be two systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$. Then $\mathcal{R}_{L/K}(M, \{s_i\}) \simeq \mathcal{R}_{L/K}(M', \{s_i'\})$ if and only if $(M, \{s_i\}) \simeq (M', \{s_i'\})$.*

Proof. By Corollary 4.4, it is enough to prove $\mathcal{R}_{L/K}F(M, \{s_i\}) \simeq \mathcal{R}_{L/K}F(M', \{s_i'\})$ if and only if $F(M, \{s_i\}) \simeq F(M', \{s_i'\})$ (with F as in Theorem 4.1). Write $(Z, (\alpha, \alpha^{\mathrm{op}})) = F(M, \{s_i\}) \oplus F(M', \{s_i'\})$ and let $E = \mathrm{End}(Z)$. Then E is a K-subalgebra of $\mathrm{End}(M \oplus M') \times \mathrm{End}(M \oplus M')^{\mathrm{op}}$, which is finite dimensional. By applying $\mathrm{T}_{(Z, (\alpha, \alpha^{\mathrm{op}}))}$ (see subsection 2.3), we reduce to showing that two 1-hermitian forms over E are isometric over $E \otimes_K L$ if and only if they are isometric over E, which is just [2, Th. 2.1]. (Note that we used the fact that transfer commutes with $\mathcal{R}_{L/K}$ in the sense of subsection 2.5.) $\qquad\square$

Corollary 5.13. *Let A be a f.d. K-algebra and let $\{\sigma_i\}_{i \in I}$ be a nonempty family of K-involutions on A. Let $(M, \{s_i\})$, $(M', \{s_i'\})$ be two systems of sesquilinear forms over $(A, \{\sigma_i\})$. If M and M' are of finite type, then $\mathcal{R}_{L/K}(M, \{s_i\}) \simeq \mathcal{R}_{L/K}(M', \{s_i'\})$ if and only if $(M, \{s_i\}) \simeq (M', \{s_i'\})$.*

To state the analogue of the injectivity of $r_{L/K} : \mathrm{W}(K) \to \mathrm{W}(L)$ for hermitian categories, we need to introduce additional notation.

An additive category $\mathscr{C}$ is called *semi-abelian* if all idempotents in $\mathscr{C}$ split. Any additive category $\mathscr{C}$ admits a *semi-abelian closure* (e.g. see [9, Th. 6.10]), namely, a semi-abelian additive category $\mathscr{C}^\circ$ equipped with an additive functor $A \mapsto A^\circ : \mathscr{C} \to \mathscr{C}^\circ$, such that the pair $(\mathscr{C}^\circ, A \mapsto A^\circ)$ is *universal*. The category $\mathscr{C}^\circ$ is unique up to equivalence and the functor $A \mapsto A^\circ$ turns out to be faithful and full. The category $\mathscr{C}^\circ$ can be realized as the category of pairs (M, e) with $M \in \mathscr{C}$ and $e \in \mathrm{End}_\mathscr{C}(M)$ an idempotent. The Hom-sets in $\mathscr{C}^\circ$ are given by $\mathrm{Hom}_{\mathscr{C}^\circ}((M, e), (M', e')) = e' \mathrm{Hom}_\mathscr{C}(M, M')e$ and the composition is the same as in $\mathscr{C}$. Finally, set $M^\circ = (M, \mathrm{id}_M)$ and $f^\circ = f$ for any object $M \in \mathscr{C}$ and any

morphism f in $\mathscr{C}$. For simplicity, we will use only this particular realization of $\mathscr{C}^\circ$. Nevertheless, the universality implies that the statements to follow hold for any semi-abelian closure.

Assume $\mathscr{C}$ admits a K-linear hermitian structure $(*, \omega)$. Then $\mathscr{C}^\circ$ is clearly a K-category, and moreover, it has a K-linear hermitian structure given by $(M, e)^* = (M^*, e^*)$ and $\omega_{(M,e)} = e^{**}\omega_M e \in \operatorname{Hom}_{\mathscr{C}^\circ}((M, e), (M^{**}, e^{**}))$. Furthermore, the functor $M \mapsto M^\circ$ is 1-hermitian and duality preserving (the isomorphism $(M^*)^\circ \to (M^\circ)^*$ being id_M), so we have a faithful and full functor $(M, s) \mapsto (M, s)^\circ = (M^\circ, s)$ from $\operatorname{Sesq}(\mathscr{C})$ to $\operatorname{Sesq}(\mathscr{C}^\circ)$. Henceforth, consider $\mathscr{C}$ (resp. $\operatorname{Sesq}(\mathscr{C})$) as a full subcategory of $\mathscr{C}^\circ$ (resp. $\operatorname{Sesq}(\mathscr{C}^\circ)$), i.e identify M° (resp. $(M, s)^\circ$) with M (resp. (M, s)).

Lemma 5.14. *Let $\mathscr{C}$, $\mathscr{C}'$ be two hermitian categories and let $F : \mathscr{C} \to \mathscr{C}'$ be an ϵ-hermitian duality preserving functor. Then:*

   (i) F extends to an ϵ-hermitian duality preserving functor $F^\circ : \mathscr{C}^\circ \to \mathscr{C}'^\circ$. If F is faithful and full, then so is F°.
   (ii) There is a 1-hermitian duality preserving functor $G : \widetilde{\operatorname{Ar}}_2(\mathscr{C})^\circ \to \widetilde{\operatorname{Ar}}_2(\mathscr{C}^\circ)$. The functor G fixes $\widetilde{\operatorname{Ar}}_2(\mathscr{C})$ and induces an equivalence of categories.

Proof. (i) Define $F^\circ(M, e) = (FM, Fe) \in \mathscr{C}'^\circ$. The rest is routine.

(ii) Let G send $((M, M', f, g), (e, e'^{\operatorname{op}})) \in \widetilde{\operatorname{Ar}}_2(\mathscr{C})^\circ$ to $((M, e), (M, e'), e'^* f e, e'^* g e)$ and any morphism to itself. The details are left to the reader. $\square$

Observe that the category $\mathscr{C}_L$ may not be semi-abelian even when $\mathscr{C}$ is. We thus set $\mathscr{C}_L^\circ := (\mathscr{C}_L)^\circ$.

Theorem 5.15. *Let $(\mathscr{C}, *, \omega)$ be a semi-abelian K-linear hermitian category such that $\dim_K \operatorname{Hom}(M, M') < \infty$ for all $M, M' \in \mathscr{C}$. Then the maps*

$$\operatorname{W}^\epsilon(\mathcal{R}_{L/K}) : \operatorname{W}^\epsilon(\mathscr{C}) \to \operatorname{W}^\epsilon(\mathscr{C}_L^\circ) \qquad and \qquad \operatorname{W}(\mathcal{R}_{L/K}) : \operatorname{W}_S(\mathscr{C}) \to \operatorname{W}_S(\mathscr{C}_L^\circ)$$

are injective.

Proof. We begin by showing that $\operatorname{W}^\epsilon(\mathcal{R}_{L/K}) : \operatorname{W}^\epsilon(\mathscr{C}) \to \operatorname{W}^\epsilon(\mathscr{C}_L^\circ)$ is injective. Let $(M, s) \in \operatorname{UH}^\epsilon(\mathscr{C})$ be such that $(M_L, s_L) \equiv 0$ in $\operatorname{W}^\epsilon(\mathscr{C}_L^\circ)$. Then there are objects $N, N' \in \mathscr{C}_L^\circ$ such that $s_L \oplus \mathbb{H}_N^\epsilon \simeq \mathbb{H}_{N'}^\epsilon$. Let $(U, h) = (M, s) \oplus (N', \mathbb{H}_{N'}^\epsilon)$, $E = \operatorname{End}_{\mathscr{C}^\circ}(U)$ and let σ be the involution induced by h on E. Also set $E_L = E \otimes_K L = \operatorname{End}_{\mathscr{C}_L^\circ}(U_L)$ and $\sigma_L = \sigma \otimes_K \operatorname{id}_L$. Subsection 2.5 implies that $\mathcal{R}_{L/K}(\operatorname{T}_{(U,h)}(M, s)) = \operatorname{T}_{(U_L, h_L)}(M_L, s_L) \equiv 0$ in $\operatorname{W}^\epsilon(E_L, \sigma_L)$, and by [2, Prp. 1.2], this means $\operatorname{T}_{(U,h)}(M, s) \equiv 0$ in $\operatorname{W}^\epsilon(E, \sigma)$ (here we need $\dim_K E < \infty$). Since $\mathscr{C}$ is semi-abelian, the map $\operatorname{T}_{(U,h)} : \mathscr{C}|_U \to \mathcal{P}(E)$ is an equivalence of categories, hence the induced map $\operatorname{W}^\epsilon(\operatorname{T}_{(U,h)}) : \operatorname{W}^\epsilon(\mathscr{C}|_U) \to \operatorname{W}(\mathcal{P}(E)) = \operatorname{W}^\epsilon(E, \sigma)$ is an isomorphism of groups. Therefore, $(M, s) \equiv 0$ in $\operatorname{W}^\epsilon(\mathscr{C}|_U)$. In particular, the same identity holds in $\operatorname{W}^\epsilon(\mathscr{C})$.

Now let $(M, s) \in \operatorname{Sesq}(\mathscr{C})$ be such that $(M_L, s_L) \equiv 0$ in $\operatorname{W}_S(\mathscr{C}_L^\circ)$. Then by Proposition 5.10, (M_L, s_L) is hyperbolic in $\mathscr{C}_L^\circ$ (but, a-priori, not in $\mathscr{C}_L$). Let

F be the functor defined in Theorem 3.2 and let J be the functor $\widetilde{\mathrm{Ar}}_2(\mathscr{C})_L \to \widetilde{\mathrm{Ar}}_2(\mathscr{C}_L)$ of Proposition 3.7. By the lemma, there is a fully faithful 1-hermitian duality preserving functor $J' := GJ^\circ : \widetilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ \to \widetilde{\mathrm{Ar}}_2(\mathscr{C}_L^\circ)$. Since (M_L, s_L) is hyperbolic in $\mathscr{C}_L^\circ$, there is $Q \in \widetilde{\mathrm{Ar}}_2(\mathscr{C}_L^\circ)$ such that $F(M_L, s_L) \simeq (Q \oplus Q^*, \mathbb{H}_Q)$. Let $Z(M, s) := (M, M, s^*\omega_M, s)$ and $Z(M_L, s_L) = (M_L, M_L, s_L^*\omega_{M_L}, s_L)$. Recall that $F(M_L, s_L) = F\mathcal{R}_{L/K}(M, s) = J\mathcal{R}_{L/K}F(M, s)$ (Proposition 3.7) and hence $Q \oplus Q^* \simeq Z(M_L, s_L) = J(Z(M, s)_L) = J'(Z(M, s)_L)$. As J' is fully faithful and its image is semi-abelian, we may assume $Q = J'H$ for some $H \in \widetilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ$. We now have $J'(H \oplus H^*, \mathbb{H}_H) = (Q \oplus Q^*, \mathbb{H}_Q) \simeq F(M_L, s_L) = J'\mathcal{R}_{L/K}F(M, s)$, hence $(H \oplus H^*, \mathbb{H}_H) \simeq \mathcal{R}_{L/K}F(M, s)$ in $\widetilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ$. In particular, $\mathcal{R}_{L/K}F(M, s) \equiv 0$ in $\mathrm{W}(\widetilde{\mathrm{Ar}}_2(\mathscr{C})_L^\circ)$. By the previous paragraph, this means $F(M, s) \equiv 0$ in $\mathrm{W}(\widetilde{\mathrm{Ar}}_2(\mathscr{C}))$ and hence, $(M, s) \equiv 0$ in $\mathrm{W}_S(\mathscr{C})$. $\qquad\square$

We also have the following weaker version of Springer's Theorem that works without assuming $\mathscr{C}$ is semi-abelian.

Theorem 5.16. *Let $(\mathscr{C}, *, \omega)$ be a K-linear hermitian category such that $\dim_K \mathrm{Hom}(M, M') < \infty$ for all $M, M' \in \mathscr{C}$. Then the map $\mathrm{W}^\epsilon(\mathcal{R}_{L/K}) : \mathrm{W}^\epsilon(\mathscr{C}) \to \mathrm{W}^\epsilon(\mathscr{C}_L)$ is injective.*

Proof. Let $(M, s) \in \mathrm{UH}^\epsilon(\mathscr{C})$ be such that $(M_L, s_L) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C}_L)$. Then there are objects N_L, N_L' such that $s_L \oplus \mathbb{H}_{N_L}^\epsilon \simeq \mathbb{H}_{N_L'}^\epsilon$. Since $\mathbb{H}_{N_L}^\epsilon = (\mathbb{H}_N^\epsilon)_L$ and $\mathbb{H}_{N_L'}^\epsilon = (\mathbb{H}_{N'}^\epsilon)_L$, we have $(s \oplus \mathbb{H}_N^\epsilon)_L \simeq (\mathbb{H}_{N'}^\epsilon)_L$. By Theorem 5.12, this means $s \oplus \mathbb{H}_N^\epsilon \simeq \mathbb{H}_{N'}^\epsilon$, hence $(M, s) \equiv 0$ in $\mathrm{W}^\epsilon(\mathscr{C})$. $\qquad\square$

5.6. **Weak Hasse Principle.** In this final subsection, we prove a version of the *weak Hasse principle* for systems of sesquilinear forms over hermitian categories. Recall that the weak Hasse principle asserts that two quadratic forms over a global field k are isometric if and only if they are isometric over all completions of k. This actually fails for systems of quadratic forms, and we refer the reader to [4] and [5] for necessary and sufficient conditions for the weak Hasse principle to hold in this case. A weak Hasse principle for *sesquilinear* forms defined over a skew field with a unitary involution was obtained in [3].

Let K be a commutative ring admitting an involution σ, and let k be the fixed ring of σ. Let $\mathscr{C}$ be an additive K-category. A hermitian structure $(*, \omega)$ on $\mathscr{C}$ is called (K, σ)-*linear* if $(fa)^* = f^*\sigma(a)$ for all $a \in K$ and any morphism f in $\mathscr{C}$. (This means that the functor $*$ is k-linear.) In this case, $\mathrm{End}(M)$ is a K-algebra for all $M \in \mathscr{C}$, and for any unimodular ϵ-hermitian form (M, s) over $\mathscr{C}$, the restriction of the involution $f \mapsto s^{-1}f^*s$ to $K \cdot \mathrm{id}_M$ is σ.

Suppose now that K is a global field of characteristic not 2 and $\mathscr{C}$ admits a nonempty family of (K, σ)-linear hermitian structures $\{*_i, \omega_i\}_{i \in I}$. For every prime spot p of k, let k_p be the completion of k at p and set $K_p = K \otimes_k k_p$, $\sigma_p = \sigma \otimes_k \mathrm{id}_{k_p}$ and $\mathscr{C}_p = \mathscr{C} \otimes_k k_p$. Then each of the hermitian structures $(*_i, \omega_i)$

gives rise to a (K_p, σ_p)-linear hermitian structure on $\mathscr{C}_p$, which we also denote by $(*_i, \omega_i)$.

Theorem 5.17. *Let K be a global field of characteristic not 2 admitting an involution σ of the second kind* with fixed field k. *Let $\mathscr{C}$ be a K-category such that $\dim_K \operatorname{Hom}(M, N) < \infty$ for all $M, N \in \mathscr{C}$, and assume there is a nonempty family $\{*_i, \omega_i\}_{i \in I}$ of (K, σ)-linear hermitian structures on $\mathscr{C}$. Then the weak Hasse principle (w.r.t. k) holds for systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$. That is, two systems of sesquilinear forms over $(\mathscr{C}, \{*_i, \omega_i\})$ are isometric if and only if they isometric after applying $\mathcal{R}_{k_p/k}$ for all p.*

Proof. By Corollary 4.4, it is enough to verify the Hasse principle (w.r.t. k) for 1-hermitian forms in the category $\mathscr{G} := \widetilde{\operatorname{Ar}}_{2I}(\mathscr{C})$. Our assumptions imply that $\mathscr{G}$ is a (K, σ)-linear category such that $\dim_K \operatorname{Hom}(Z, Z') < \infty$ for all $Z, Z' \in \mathscr{G}$. We now use the ideas developed in [3, §9].

Let (Z, h), (Z', h') be two unimodular 1-hermitian forms over $\mathscr{G}$ such that $\mathcal{R}_{k_p/k}(Z, h) \simeq \mathcal{R}_{k_p/k}(Z', h')$ for all p. This is implies that $Z \cong Z'$,[4] so we may assume $Z = Z'$.

Fix a 1-hermitian form h_0 on Z and let τ be the involution induced by h_0 on $E := \operatorname{End}(Z)$ (i.e. $\tau(x) = h_0^{-1} x^* h_0$). There is an equivalence relation on the elements of E defined by $x \sim y \iff$ there exists invertible $z \in E$ such that $x = zy\tau(z)$. Let $H(\tau, E^\times)$ be the set of equivalence classes of invertible elements $x \in E^\times$ for which $x = \tau(x)$. In the same manner as in [3, Th. 5.1], we see that there is a one-to-one correspondence between isometry classes of unimodular 1-hermitian forms on Z and elements $H(\tau, E^\times)$. It is given by $(Z, t) \mapsto h_0^{-1} t$.

Applying the same arguing to $Z_p = \mathcal{R}_{k_p/k} Z \in \mathscr{G}_p$, we see that the weak Hasse principle is equivalent to the injectivity of the standard map

$$\Phi : H(\tau, E^\times) \to \prod_p H(\tau_p, E_p^\times)$$

where $E_p = \operatorname{End}(Z_p) = E \otimes_k k_p$ and $\tau_p = \tau \otimes_k \operatorname{id}_{k_p}$. Observe that since $\mathscr{G}$ is (K, σ)-linear, τ is a unitary involution (and in fact, $\tau|_K = \sigma$). By [3, §9], this means that Φ is injective, hence the weak Hasse principal holds. $\square$

Corollary 5.18. *Let K be a global field of characteristic not 2 admitting an involution σ of the second kind* with fixed field k. *Let A be a f.d. K-algebra*

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[4] For any field extension L/k, $Z \cong Z' \iff Z_L \cong Z_L'$. Here is an ad-hoc proof: By applying $\operatorname{Hom}_{\mathscr{C}}(Z \oplus Z', \_)$, we may assume Z and Z' are f.g. projective right modules over $R := \operatorname{End}(Z \oplus Z')$, which is a f.d. k-algebra. Let J be the Jacobson radical of R. By tensoring with R/J, we may assume R is semisimple. Let $\{V_i\}_i$ be a complete list of the simple right R-modules and write $(V_i)_L = \bigoplus_j W_{ij}^{n_{ij}}$ with $\{W_{ij}\}_j$ being pairwise non-isomorphic indecomposable R_L-modules. The modules $\{W_{ij}\}_{i,j}$ are pairwise non-isomorphic because W_{ij} and $W_{i'j'}$ have different annihilators in R_L if $i \neq i'$. Assume $Z_L \cong Z_L'$ and write $Z \cong \bigoplus_i V_i^{m_i}$, $Z' \cong \bigoplus_i V_i^{m_i'}$. Then $\bigoplus_{i,j} W_{ij}^{m_i n_{ij}} \cong Z_L \cong Z_L' \cong \bigoplus_{i,j} W_{ij}^{m_i' n_{ij}}$. By the Krull-Schmidt Theorem (see for instance [15, pp. 237]), we have $m_i n_{ij} = m_i' n_{ij}$ for all i, j, hence $m_i = m_i'$ and $Z \cong Z'$.

admitting a nonempty family of involutions $\{\sigma_i\}_{i\in I}$ such that $\sigma_i|_K = \sigma$. Then the weak Hasse principle (w.r.t. k) holds for systems of sesquilinear forms over $(A, \{\sigma_i\})$.

Acknowledgements

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